JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **34**, No. 3, August 2021
http://dx.doi.org/10.14403/jcms.2021.34.3.253

THE ENUMERATION OF INVOLUTIONS OF DOUBLY ALTERNATING BAXTER PERMUTATIONS

SOOK MIN

ABSTRACT. In this paper, we provide a recursive formula for the number of involutions of doubly alternating Baxter permutations in S_n.

1. Introduction

We begin with some definitions and notations. Let S_n denote the symmetric group of all permutations of $[n] = \{1, 2, ..., n\}$. A *Baxter permutation* is exactly a permutation $\pi = a_1 a_2 \cdots a_n$ in S_n that satisfies the following two conditions: for every $1 \le i < j < k < l \le n$,

$$\begin{aligned} \text{if } a_i + 1 &= a_l \text{ and } a_l < a_j \text{ then } a_k > a_l, \text{ and} \\ \text{if } a_l + 1 &= a_i \text{ and } a_i < a_k \text{ then } a_j > a_i. \end{aligned}$$

For example, 2413 and 3142 are the only permutations on four elements which are not Baxter permutations. Baxter permutations, named by Boyce, first arose in attempts to prove the "commuting function" conjecture [1]. Chung, Graham, Hoggatt and Kleiman [2] enumerated analytically the family of Baxter permutations.

The *descent set* of $\pi = a_1 a_2 \cdots a_n$ in S_n, $Des(\pi)$, is the set of integer i with $1 \le i < n$ such that $a_i > a_{i+1}$. More precisely, $Des(\pi) = \{i | a_i > a_{i+1}\}$. A permutation $\pi = a_1 a_2 \cdots a_n$ in S_n is called *alternating* permutation if $a_1 < a_2 > a_3 < a_4 > \cdots$, that is to say, $Des(\pi)$ happens at even index. A permutation $\pi = a_1 a_2 \cdots a_n$ in S_n is called *reverse alternating* permutation if $a_1 > a_2 < a_3 > a_4 < \cdots$, that is to say, $Des(\pi)$ happens at odd index.

Received May 25, 2021; Accepted July 22, 2021.
2010 Mathematics Subject Classification: Primary 05A05; Secondary 05A99.
Key words and phrases: involution, alternating, reverse alternating, Baxter permutation.

Cori, Dulucq and Viennot [3] proved that alternating Baxter permutations of length $2n$ and $2n+1$ are enumerated by C_n^2 and C_nC_{n+1}, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number.

A permutation π in S_n is called *doubly alternating* if both π and π^{-1} are alternating. Let B_n denote the set of all doubly alternating Baxter permutations of length n and let RB_n denote the set of all doubly reverse alternating Baxter permutations of length n. Guibert and Linusson [4] and Min and Park [6] showed that $|B_{2n+\epsilon}| = C_n$, where $\epsilon = 0$ or 1. A permutation π in S_n is called an *involution* if $\pi = \pi^{-1}$. Let $B_n(I)$ denote the set of all involutions in B_n and let $RB_n(I)$ denote the set of all involutions in RB_n. The purpose of this paper is to provide a recursive formula for the cardinality of the set $B_n(I)$.

2. Main theorem

In this section, we will prove the relation between two sets $B_{2n+1}(I)$ and $RB_{2n}(I)$ in Lemma 2.5, and then prove the purpose of this paper in Theorem 2.7.

LEMMA 2.1. ([6, Theorem 4.1]) If $\pi = a_1a_2\cdots a_{2n} \in B_{2n}$ then

$$\pi = (2k+1)\, a_2\, a_3\, \cdots\, a_{2n-2k-1}(2n)\, a_{2n-2k+1} \cdots\, a_{2n}$$

such that $\{a_{2n-2k+1}, \ldots, a_{2n}\} = \{1, 2, \ldots, 2k\}$ where $0 \le k \le n-1$.

LEMMA 2.2. ([4, Corollary 8], [6, Corollary 4.3]) If $\pi = a_1a_2\cdots a_{2n+1} \in RB_{2n+1}$ then $a_{2n+1} = 2n+1$ and $\sigma = a_1a_2\cdots a_{2n} \in RB_{2n}$.

For example, 21435, 42315 are the only permutations on five elements satisfying Lemma 2.2.

LEMMA 2.3. ([6, Corollary 4.5]) If $\pi = a_1a_2\cdots a_{2n} \in RB_{2n}$ then

$$\pi = (2k)\, a_2 \cdots a_{2k-1} 1\, a_{2k+1} \cdots\, a_{2n}$$

such that $\{2k,\, a_2, \ldots, a_{2k-1}, 1\} = \{1, 2, \ldots, 2k\}$, where $1 \le k \le n$.

LEMMA 2.4. ([6, Corollary 4.7]) If $\pi = a_1a_2\cdots a_{2n+1} \in B_{2n+1}$ then $a_1 = 1$ and $\sigma = (a_2-1)(a_3-1)\cdots(a_{2n+1}-1) \in RB_{2n}$.

For example, 15342, 13254 are the only permutations on five elements satisfying Lemma 2.4.

LEMMA 2.5. $|B_{2n+1}(I)| = |RB_{2n}(I)|$.

$B_7(I)$	bijection	$RB_6(I)$
1735462	$\longleftrightarrow$	624351
1756342	$\longleftrightarrow$	645231
1534276	$\longleftrightarrow$	423165
1325476	$\longleftrightarrow$	214365
1327564	$\longleftrightarrow$	216453

FIGURE 1. A bijection between $|B_7(I)|$ and $|RB_6(I)|$.

Proof. Let $\sigma \in RB_{2n}(I)$. If we put $\pi = 1\,(\sigma_1+1)(\sigma_2+1)\cdots(\sigma_{2n}+1)$, then it is also an involution of Baxter permutations of length $2n+1$ with $Des(\pi) = Des(\pi^{-1}) = \{2,4,\ldots,2n\}$, since 1 is the smallest element. That is to say, $\pi \in B_{2n+1}(I)$. From Lemma 2.4, the above π's are the only doubly alternating Baxter permutations of length $2n+1$. □

EXAMPLE 2.6. *For $n = 3$, we give a bijection list for $5 = |B_7(I)| = |RB_6(I)|$: see Figure 1.*

Then we provide our main theorem.

THEOREM 2.7. *If $b_n = |B_n(I)|$ then its recursive formula is*

$$b_{2n-1} = \sum_{k=1}^{n-1} b_{2k-2} \cdot b_{2n-2k-1} \ (n \geq 2),$$

$$b_{2n} = b_{2n-1} + \sum_{k=\lceil \frac{n}{2} \rceil}^{n-1} b_{2n-2k-1} \cdot b_{4k-2n} \ (n \geq 3).$$

Note that we have $b_0 = 1$ by considering the empty permutation and $b_1 = b_2 = 1$, by a direct computation.

Proof. First we prove the second formula. If $\pi \in B_{2n}(I)$, then by Lemma 2.1 we can write $\pi = (2k+1)\,a_2\cdots a_{2n-2k-1}\,(2n)\,a_{2n-2k+1}\cdots a_{2n}$ such that $\{a_{2n-2k+1},\ldots,a_{2n}\} = \{1,2,\ldots,2k\}$ where $0 \leq k \leq n-1$.

If $k = 0$ then $\pi = 1a_2\cdots a_{2n-1}\ (2n)$ and $(a_2-1)(a_3-1)\cdots(a_{2n-1}-1) \in RB_{2n-2}(I)$. So the number of such permutations is $|RB_{2n-2}(I)| = |B_{2n-1}(I)| = b_{2n-1}$ by Lemma 2.5.

Now suppose $k \neq 0$.

(Case 1) $2n-2k+1 > 2k+1$: If $\pi^{-1}(1) = j$ then j is in the set $\{2n-2k+1, 2n-2k+2, \ldots, 2n\}$. Since $2n-2k+1 > 2k+1$, $\pi^{-1}(1)$ can not be $2k+1$. This is a contradiction, since $\pi \in B_{2n}(I)$. So we assume $2n-2k+1 \leq 2k+1$.

(Case 2) $2n-2k+1 \leq 2k+1 (\Leftrightarrow \; n \leq 2k)$: If $n=2$ then $b_4 = 2$, since 1324 and 3412 are the only permutations in $B_4(I)$. Now suppose $3 \leq n \leq 2k$. We can write π as follows:

$$\pi = (2k+1)a_2 \cdots a_{2n-2k-1}(2n)a_{2n-2k+1} \cdots a_n \cdots a_{2k+1} \cdots a_{2n}$$

Put

$$\pi = \pi_1 \pi_2 \pi_3,$$

where

$$\begin{aligned} \pi_1 &= (2k+1)\, a_2 \cdots a_{2n-2k-1}(2n), \\ \pi_2 &= a_{2n-2k+1} \cdots a_n \cdots a_{2k}, \\ \pi_3 &= a_{2k+1} \cdots a_{2n}. \end{aligned}$$

If we consider the permutation $\pi_1^* = 1(a_2 - 2k) \cdots (a_{2n-2k-1} - 2k)(2n - 2k)$ corresponding to π_1, then $\pi_1^* \in B_{2n-2k}(I)$ and

$$(a_2 - 2k - 1)(a_3 - 2k - 1) \cdots (a_{2n-2k-1} - 2k - 1) \in RB_{2n-2k-2}(I).$$

So $|RB_{2n-2k-2}(I)| = |B_{2n-2k-1}(I)| = b_{2n-2k-1}$ by Lemma 2.5. Since π is an involution, $\pi_1^* = \pi_3$. If we consider the permutation π_2^* corresponding to π_2:

$$\pi_2^* = c_{2n-2k+1} \cdots c_{2k},$$

where $c_i = a_i - (2n-2k)$, then $\pi_2^* \in B_{4k-2n}(I)$ and $|B_{4k-2n}(I)| = b_{4k-2n}$. Thus we have the recursive formula, for $n \geq 3$,

$$b_{2n} = b_{2n-1} + \sum_{k=\lceil \frac{n}{2} \rceil}^{n-1} b_{2n-2k-1} \cdot b_{4k-2n}.$$

Now we prove the first formula. Note that $b_{2n-1} = |B_{2n-1}(I)| = |RB_{2n-2}(I)|$, by Lemma 2.5. If $\sigma \in RB_{2n-2}(I)$, then by Lemma 2.3 we can write $\sigma = (2k)\, a_2 \cdots a_{2k-1}\, 1\, a_{2k+1} \cdots a_{2n-2}$ such that $\{2k, a_2, \ldots, a_{2k-1}, 1\} = \{1, 2, \ldots, 2k\}$ where $1 \leq k \leq n-1$.

Put $\sigma = \sigma_1 \sigma_2$, where

$$\begin{aligned} \sigma_1 &= (2k)\, a_2 \cdots a_{2k-1} 1, \\ \sigma_2 &= a_{2k+1} \cdots a_{2n-2}. \end{aligned}$$

If we consider the permutation $\sigma_1^* = (a_2 - 1)(a_3 - 1) \cdots (a_{2k-1} - 1)$ corresponding to σ_1, then $\sigma_1^* \in B_{2k-2}(I)$ and $|B_{2k-2}(I)| = b_{2k-2}$. If we consider the permutation σ_2^* corresponding to σ_2:

$$\sigma_2^* = c_{2k+1} \cdots c_{2n-2},$$

where $c_i = a_i - 2k$, then $\sigma_2^* \in RB_{2n-2k-2}(I)$ and $|RB_{2n-2k-2}(I)| = |B_{2n-2k-1}(I)| = b_{2n-2k-1}$. So we have the recursive formula, for $n \geq 2$,

$$b_{2n-1} = |RB_{2n-2}(I)| = \sum_{k=1}^{n-1} b_{2k-2} \cdot b_{2n-2k-1}.$$

□

The first few values of $b_n (n = 0, 1, 2, 3, ...)$ are $1, 1, 1, 1, 2, 2, 3, 5, 8, 12, 16, 32, 44, 84, 105, 231, 292, 636, 768, 1792, 2166, 5080, 6012, 14592, 17234, 42198, 49336, ...$ The sequence does not match anything in the OEIS(Online Encyclopedia of Integer Sequences)(2021.5.25.)

The material in this paper is from the author's Ph.D. thesis [5].

References

[1] G. Baxter, *On fixed points of the composite of commuting functions,* Proc. Amer. Math. Soc., **15** (1964), 851-855.

[2] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., and M. Kleiman, *The number of Baxter permutations,* J. Combin. Theory Ser. A., **24** (1978), 382-394.

[3] B. R. Cori, S. Dulucq, and G. Viennot, *Shuffle of parenthesis systems and Baxter permutations,* J. Combin. Theory Ser. A., **43** (1986), 1-22.

[4] O. Guibert and S. Linusson, *Doubly alternating Baxter permutations are Catalan,* Discrete Math., **217** (2000), 157-166.

[5] S. Min, *On the essential sets of the Baxter and pseudoBaxter permutations,* PhD thesis, Yonsei University, 2002.

[6] S. Min and S. Park, *The enumeration of doubly alternating Baxter permutations,* J. Korean Math. Soc., **43** (2006), 553-561.

*
Division of Software
Yonsei University
Wonju, Republic of Korea
E-mail: sookmin@yonsei.ac.kr